A PARALLELOGRAM PROBLEM

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A *diagonal angle* of a parallelogram is an angle bounded by a side and a diagonal. A parallelogram has eight diagonal angles, which break into four pairs of congruent angles. In our discussion, we take $ABA'B'$ to be a parallelogram with diagonal angle measures α, β, α', and β', and various segment lengths as shown in Figure 1.

Exercise. *Given two of the angle measures α, β, α', and β', determine the remaining two.*

There are six ways to choose the pairs of given angles, leading to four cases that are easy to solve and two that are not-so-easy.

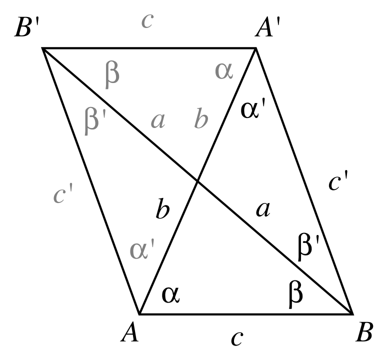

FIGURE 1. Labeled Parallelogram

1. Four Easy Cases

Here, we'll cover the following situations:

Case 1: α , β given; α', β' sought.
Case 2: α', β' given; α , β sought.
Case 3: α, α' given; β, β' sought.
Case 4: β, β' given; α, α' sought.

We see easily that Case 1 and Case 2 are analogous: given two angles in one of the quadrants of the parallelogram, we seek the angles in another quadrant. (Symbolically, we simply apply the change of variables $\alpha \leftrightarrow \alpha'$ and $\beta \leftrightarrow \beta'$.) Case 3 and Case 4 are also analogous: given two angles making up one corner of the parallelogram, we seek the angles making up another corner. (Symbolically, $\alpha \leftrightarrow \beta$ and $\alpha' \leftrightarrow \beta'$.) In point of fact, Case 3 and Case 4 are also analogous to Case 1: the given angles appear in quadrants of a secondary parallelogram that arises when four copies of the original parallelogram are joined together. Thus, we may focus on solving Case 1.

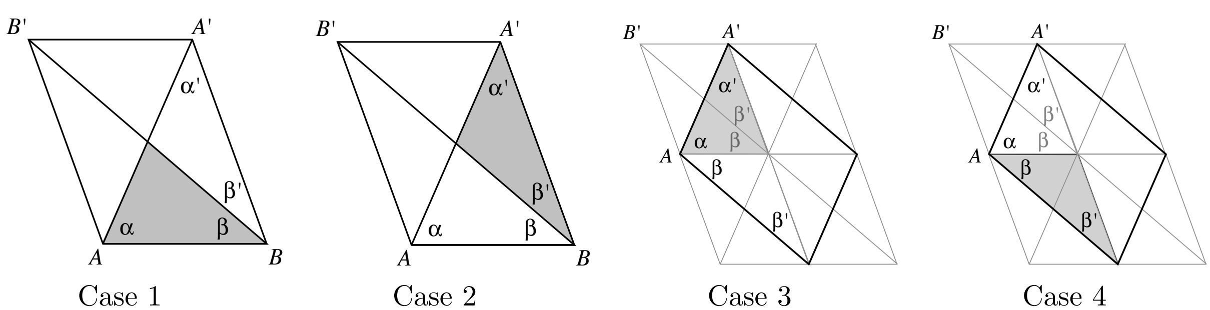

FIGURE 2. The Equivalence of the Four Easy Cases.

Two straightforward trigonometric relations will come in handy. The first relates the side of a triangle to its adjacent elements; for instance,

$$c = a\cos\beta + b\cos\alpha \tag{1}$$

and the second (which is provable by the Law of Cosines) relates the parallelogram's sides:

$$2a^2 + 2b^2 = c^2 + c'^2 \tag{2}$$

Finally, the Law of Sines guarantees that

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin(\pi - \alpha - \beta)}{c} \tag{3}$$

Convenient scaling of the diagram allows us to assume that the Law of Sines ratio is 1. That is,

$$a = \sin\alpha \qquad b = \sin\beta \qquad c = \sin(\pi - \alpha - \beta) = \sin(\alpha + \beta) \tag{4}$$

At this point, we may proceed directly to our computations: Substituting back through Equations (2) and (1), we arrive at the following results. (To conserve space, we represent "$\sin\theta$" by "$[\theta]$".)

$$\cos\alpha' = \frac{b - a\cos(\alpha+\beta)}{c'} = \frac{[\beta] - [\alpha]\cos(\alpha+\beta)}{\sqrt{[\alpha]^2 + [\beta]^2 - 2[\alpha][\beta]\cos(\alpha+\beta)}}$$

$$\cos\beta' = \frac{a - b\cos(\alpha+\beta)}{c'} = \frac{[\alpha] - [\beta]\cos(\alpha+\beta)}{\sqrt{[\alpha]^2 + [\beta]^2 - 2[\alpha][\beta]\cos(\alpha+\beta)}}$$

Note: Via Equation (2), we get the implicit relation "$c'^2 = [\alpha]^2 + [\beta]^2 - 2[\alpha][\beta]\cos(\alpha+\beta)$". Because our side lengths are non-negative we are allowed to invoke the inverse function expressing c' as an unsigned square root. Likewise, our angle measures being bounded by 0 and π allow us to apply the inverse cosine function to the above results to obtain

$$\alpha' = \operatorname{acos}\left(\frac{[\beta] - [\alpha]\cos(\alpha+\beta)}{\sqrt{[\alpha]^2 + [\beta]^2 - 2[\alpha][\beta]\cos(\alpha+\beta)}}\right)$$

$$\beta' = \operatorname{acos}\left(\frac{[\alpha] - [\beta]\cos(\alpha+\beta)}{\sqrt{[\alpha]^2 + [\beta]^2 - 2[\alpha][\beta]\cos(\alpha+\beta)}}\right)$$

For completeness, here are the results for the analogous Cases:

Case 2:

$$\alpha = \operatorname{acos}\left(\frac{[\beta'] - [\alpha']\cos(\alpha'+\beta')}{\sqrt{[\alpha']^2 + [\beta']^2 - 2[\alpha'][\beta']\cos(\alpha'+\beta')}}\right) \qquad \beta = \operatorname{acos}\left(\frac{[\alpha'] - [\beta']\cos(\alpha'+\beta')}{\sqrt{[\alpha']^2 + [\beta']^2 - 2[\alpha'][\beta']\cos(\alpha'+\beta')}}\right)$$

Case 3:

$$\beta = \operatorname{acos}\left(\frac{[\alpha'] - [\alpha]\cos(\alpha+\alpha')}{\sqrt{[\alpha]^2 + [\alpha']^2 - 2[\alpha][\alpha']\cos(\alpha+\alpha')}}\right) \qquad \beta' = \operatorname{acos}\left(\frac{[\alpha] - [\alpha']\cos(\alpha+\alpha')}{\sqrt{[\alpha]^2 + [\alpha']^2 - 2[\alpha][\alpha']\cos(\alpha+\alpha')}}\right)$$

Case 4:

$$\alpha = \operatorname{acos}\left(\frac{[\beta'] - [\beta]\cos(\beta+\beta')}{\sqrt{[\beta]^2 + [\beta']^2 - 2[\beta][\beta']\cos(\beta+\beta')}}\right) \qquad \alpha' = \operatorname{acos}\left(\frac{[\beta] - [\beta']\cos(\beta+\beta')}{\sqrt{[\beta]^2 + [\beta']^2 - 2[\beta][\beta']\cos(\beta+\beta')}}\right)$$

2. Two Not-So-Easy Cases

Here, we'll cover the situations in which the given angles appear in a kind of "pinwheel" arrangement:

Case 5: α , β' given; α', β sought.
Case 6: α', β given; α , β' sought.

Figure 3 shows these Cases to be analogous —symbolically, we apply the change of variables $\alpha \leftrightarrow \beta$ and $\alpha' \leftrightarrow \beta'$— so we need only consider Case 5.

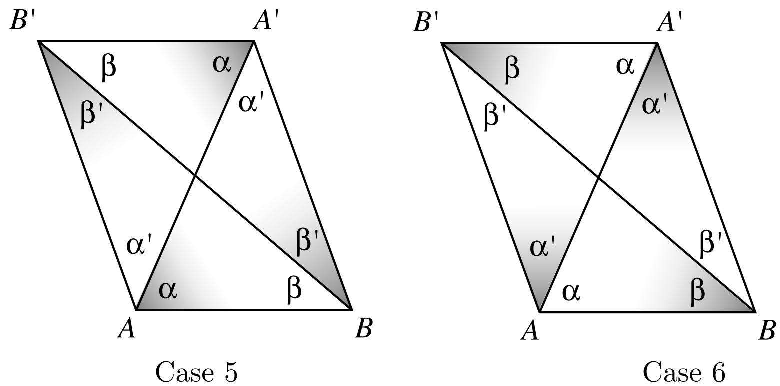

Figure 3. The Equivalence of the Two Not-So-Easy Cases.

Before we begin, we observe that this Case has a built-in ambiguity, unlike the first four Cases which each had a unique solution. The "secondary parallelogram" appearing in Cases 3 and 4 of Figure 2 has the same $\alpha\beta'$ pinwheel arrangement as the original figure, yet the angles α' and β are differently positioned relative to that arrangement. Thus, for any given angle measures α and β', we can find (at least) *two* solutions to the parallelogram problem. We'll see this fact echoed naturally in our symbolic solutions, where we'll find that there are *exactly* two possible parallelograms.

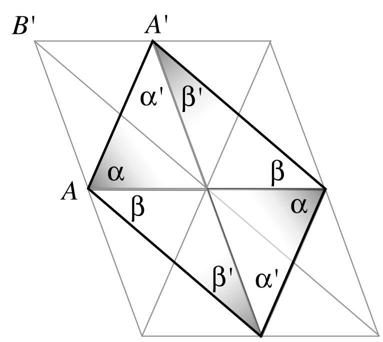

Figure 4. The Secondary Solution.

Our analysis is adapted from that of Matthew Fahrenbacher, who presented this problem in the first place.

We begin with an observation based the Law of Sines applied to the two "quadrant" triangles of the parallelogram:

$$\frac{\sin\alpha}{\sin\beta} = \frac{a}{b} = \frac{\sin\alpha'}{\sin\beta'} \tag{5}$$

Hence

$$\sin\alpha\sin\beta' = \sin\alpha'\sin\beta \tag{6}$$

Noting that $\alpha+\beta+\alpha'+\beta' = \pi$ (as the angle sum of $\triangle ABA'$, for instance), we can replace β in Equation (6) with $\pi-\alpha-\beta'-\alpha'$ and manipulate as follows:

$$\begin{aligned}
\sin\alpha\sin\beta' &= \sin\alpha'\sin\beta \\
&= \sin\alpha'\sin(\pi-\alpha-\beta'-\alpha') \\
&= \sin\alpha'\sin(\alpha+\beta'+\alpha') \\
&= \frac{1}{2}\left[\cos((\alpha+\beta'+\alpha')-\alpha') - \cos((\alpha+\beta'+\alpha')+\alpha')\right] \\
&= \frac{1}{2}\left[\cos(\alpha+\beta') - \cos(\alpha+\beta'+2\alpha')\right]
\end{aligned}$$

so that we have one of our sought angles implicitly related to our given angles:

$$\cos(\alpha+\beta'+2\alpha') = \cos(\alpha+\beta') - 2\sin\alpha\sin\beta' = 2\cos(\alpha+\beta') - \cos(\alpha-\beta') \tag{7}$$

Likewise, replacing α' in Equation (6) with $\pi-\alpha-\beta'-\beta$ yields

$$\cos(\alpha+\beta'+2\beta\ ) = 2\cos(\alpha+\beta') - \cos(\alpha-\beta') \tag{8}$$

Because Equations (7) and (8) are identical (except for the appearance of α' or β, respectively), the angle measures α' and β are not inherently distinguishable. This matches the geometric ambiguity we mentioned earlier.

Now, since either (or both) of the angle sums $\alpha+\beta'+2\alpha'$ (call this "ϕ") and $\alpha+\beta'+2\beta$ (call this "ψ") threatens to be larger than π, we cannot simply apply the inverse cosine to each side of Equations (7) and (8). To understand our situation better, we make a few observations about these angle sums.

Comparing them to partial angle sums from triangles in our parallelogram, we have

$$\begin{aligned}
0 \le \alpha+\beta'+2\alpha' &= (\alpha+\alpha') + (\beta'+\alpha') \le \pi+\pi = 2\pi \qquad \Longrightarrow \qquad 0 \le \phi \le 2\pi \\
0 \le \alpha+\beta'+2\beta\ &= (\alpha+\beta\ ) + (\beta'+\beta\ ) \le \pi+\pi = 2\pi \qquad \Longrightarrow \qquad 0 \le \psi \le 2\pi
\end{aligned}$$

and also, we have

$$(\alpha+\beta'+2\alpha') + (\alpha+\beta'+2\beta) = 2(\alpha+\beta+\alpha'+\beta') = 2\pi \quad \Longrightarrow \quad \phi+\psi = 2\pi$$

Therefore, ϕ and ψ are precisely the two solutions for θ in the equation

$$\cos\theta = 2\cos(\alpha+\beta') - \cos(\alpha-\beta')$$

in the range $0 \le \theta \le 2\pi$, although we don't know which is which. This implies that

$$\{\alpha',\beta\} = \left\{\frac{1}{2}\left(\operatorname{acos} x - \alpha - \beta'\right), \frac{1}{2}\left(2\pi - \operatorname{acos} x - \alpha - \beta'\right)\right\}, \text{ with } \ x = 2\cos(\alpha+\beta') - \cos(\alpha-\beta')$$

where, again, we don't know which angle measure is which.

For completeness, the solution to Case 6 is

$$\{\alpha,\beta'\} = \left\{\frac{1}{2}\left(\operatorname{acos} x' - \alpha' - \beta\right), \frac{1}{2}\left(2\pi - \operatorname{acos} x' - \alpha' - \beta\right)\right\}, \text{ with } \ x' = 2\cos(\alpha'+\beta) - \cos(\alpha'-\beta)$$

which concludes our analysis.