Calculus-free Derivatives of Sine and Cosine

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Wrap a square smoothly around a cylinder, and the curved images of the square's diagonals will trace out helices, with the (original, uncurved) diagonals tangent to those curves. Projecting these elements into appropriate planes reveals an intuitive, geometric development of the formulas for the derivatives of sine and cosine, with no quotient of vanishing differences required. Indeed, a single —almost-"Behold!"-worthy— diagram and a short notational summary (see the last page of this note) give everything away.

The setup. Consider a cylinder ("C") of radius 1 and a unit square ("S"), with a pair of S's edges parallel to C's axis, and with S's center ("P") a point of tangency with C's surface. We may assume that the cylinder's axis coincides with the z axis; we may also assume that, measuring along the surface of the cylinder, the (straight-line) distance from P to the xy-plane is equal to the (arc-of-circle) distance from P to the zx-plane. That is, we take P to have coordinates $(\cos\theta_0, \sin\theta_0, \theta_0)$ for some θ_0, where cos() and sin() take *radian* arguments.

Wrapping S around C causes the image of an extended diagonal of S to trace out the helix ("H") containing all (and only) points of the form $(\cos\theta, \sin\theta, \theta)$. Moreover, the original, uncurved diagonal segment of S determines a vector, $\mathbf{v} :=< v_x, v_y, v_z >$, tangent to H at P; taking $v_z = 1$, we assure that the vector conveniently points "forward" (that is, in the direction of increasing θ) along the helix.

Most importantly, *the projection of H into each coordinate plane has the corresponding projection of $\mathbf{v}$ as a tangent vector at the projection of P.*

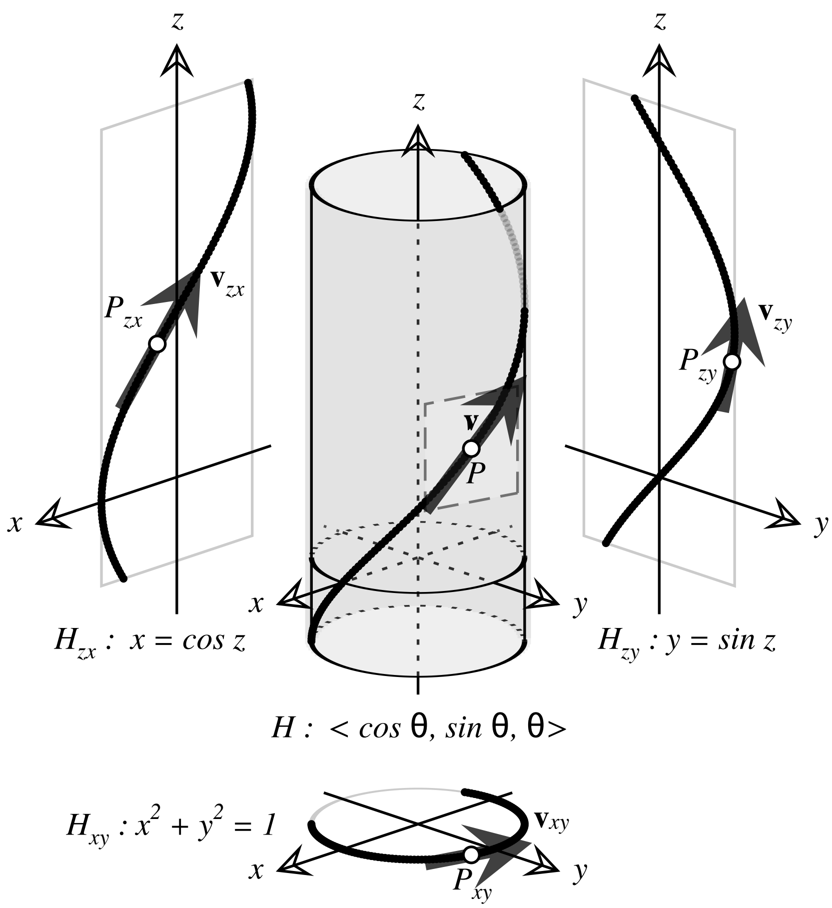
The fun begins. The projection ("H_{xy}") of H along the axis of the cylinder and into the xy-plane is the unit circle centered at the origin; the corresponding projection, $\mathbf{v}_{xy} :=< v_x, v_y, 0 >$, of $\mathbf{v}$ is tangent to H_{xy} at $P_{xy} := (\cos\theta_0, \sin\theta_0, 0)$. Elementary geometry dictates that $\mathbf{v}_{xy}$ must be perpendicular to the radius joining P_{xy} to the origin. Consequently, since $\mathbf{v}_{xy}$ is a unit vector (it is congruent to an edge of the square) and is pointing in the direction of increasing θ (just like $\mathbf{v}$), we have $v_x = -\sin\theta_0$ and $v_y = \cos\theta_0$, so that $\mathbf{v} =< -\sin\theta_0, \cos\theta_0, 1 >$.

In the zy-plane, the projection ("H_{zy}"), of H is the graph of the relation $y = \sin z$. The projection, $\mathbf{v}_{zy}$, of $\mathbf{v}$ is tangent to this curve at P_{zy}. Therefore, at the point $(z_0, y_0) = (\theta_0, \sin\theta_0) = (z_0, \sin z_0)$, the derivative of sine —*by conceptual definition*, the "change-in-y over change-in-z" slope of the line tangent to the graph— is $v_y/v_z = \cos\theta_0/1 = \cos\theta_0 = \cos z_0$.

Likewise, examining the projection of H into the zx-plane, we have that the derivative of cosine —the "change-in-x over change-in-z" slope of the line tangent to $x = \cos z$— at point $(z_0, \cos z_0)$ is $v_x/v_z = -\sin z_0$.

The derivative formulas are thereby established.

Remarks Sinusoids now join conic sections and a few other examples in the list of curves whose tangent line behavior can be revealed —and perhaps should be introduced— without the sophisticated machinery of Differential Calculus. Familiarity with these behaviors may better prepare a student to face that machinery when the time comes.



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\begin{array}{rcl}
\mathbf{v} & := & < v_x, v_y, 1 > \\
H_{xy} : \mathbf{v}_{xy} \perp \overrightarrow{OP_{xy}} & \Longrightarrow & \mathbf{v}_{xy} \;=\; < \; -\sin\theta_0, \;\; \cos\theta_0, \;\; 0 \; > \\
 & \Longrightarrow & \left\{ \begin{array}{lcl} \mathbf{v}_{zx} & = & < \; -\sin\theta_0, \;\; 0, \;\; 1 \; > \\ \mathbf{v}_{zy} & = & < \; 0, \;\; \cos\theta_0, \;\; 1 \; > \end{array} \right. \\
 & \Longrightarrow & \left\{ \begin{array}{l} H_{zx} : \dfrac{d}{dz} \cos z \Big|_{z=\theta_0} \overset{def}{=} \dfrac{\Delta x}{\Delta z} = \dfrac{-\sin\theta_0}{1} = -\sin\theta_0 \\ \\ H_{zy} : \dfrac{d}{dz} \sin z \Big|_{z=\theta_0} \overset{def}{=} \dfrac{\Delta y}{\Delta z} = \dfrac{\cos\theta_0}{1} = \cos\theta_0 \end{array} \right.
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